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Higher Order Differential Lift Equations over Manifolds

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Let M be a differentiable manifold modeled on a Banach space over K = R or C. Let $T^k(M)$ be the kth iterated tangential extension of M, and let ${}^kM$ be the kth Bowman (= restricted tangential) extension of M. It is shown that there is an embedding φ_k: ${}^kM \to T^k(M)$, and that such embeddings constitute a natural transformation of functors. Let Q be a subset/submanifold in $T^k(M)$, and let V: $Q \to T(Q)$ be a differentiable vector field. Call V *k-suitable* if every K-curve g in Q satisfying $g' = V \circ g$ has the form $g = f^{[k]}$, where $f^{[k]}$ denotes the kth iterated differential lift of a K-curve f in M. It is shown that V is k-suitable if and only if: (a) $Q = \varphi_k(\overline{Q})$, where $\overline{Q}$ is a subset/submanifold in ${}^kM$, and (b) $V = T(\varphi_k) \circ \overline{V} \circ \varphi_k^{-1}$, where $\overline{V}$: $\overline{Q} \to T(\overline{Q})$ is k-suitable relative to restricted tangential K-curve lifts $f^{(k)}$. Interpretive consequences for motion problems are discussed.

INTRODUCTION

There are two tangential resolutions for a differentiable manifold M modeled on a Banach space over the scalar field K (=R or C). One is the familiar *full* tangential resolution $(T^k(M), \pi_k)_{k\geq 0}$, where $T^0(M) = M$ and where π_k: $T^{k+1}(M) = T(T^k(M)) \to T^k(M)$ is the standard tangent bundle projection. The other is the Bowman (=*restricted* tangential) resolution $({}^kM, \pi^k, {}_kI, {}_k\pi)_{k\geq 0}$. [See Bowman (1970a,b) for the original presentation of the latter idea over general manifolds. See Bowman and Pond (1975) for a treatment when $M = G$ is a differentiable group, especially when G is a classical group of continuous linear invertible transformations on a Banach space.] Each context has its own notion of successive differential lifts of K-curves f lying in M (which we denote, respectively, by $f^{[k]}$ and $f^{(k)}$). Moreover, both $T^k(\cdot)$ and ${}^k(\cdot)$ are functors in the category of manifolds.

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In Pond (1997) it is shown, both from general theory and from numerous examples, that the full tangential context $T^k(M)$ is *not* the appropriate framework for consideration of higher order differential lift equations defined by vector fields. In this paper:

(1) A rigorous embedding linkage is established from the $^k(\cdot)$ context to the $T^k(\cdot)$ context.

(2) Subject to (1), it is shown that *every* differential lift equation formulated in the $T^k(\cdot)$ context that can reasonably be said to be of higher order can be formulated and fully treated in the (simpler) $^k(\cdot)$ context.

While this paper is concerned with structure-in-the-large rather than with examples (the latter occur in a serious way over differentiable groups and over zero-set manifolds), generic examples are presented to show that the $^k(\cdot)$ context "makes sense" from the viewpoint of physical problems, whereas the $T^k(\cdot)$ context does not.

Before beginning the development outlined above, there is a technical difficulty concerning embeddings in general Banach manifolds that does not arise in the finite-dimensional context. For instance, if A and B are Banach spaces with A a Banach subspace of B, there is no general reason to suppose that the inclusion map i: $A \to B$ is an embedding map. Specifically, there is no reason to suppose that every intrinsically differentiable K-valued function on an open set in A is locally expressible as the restriction of differentiable functions on open sets in B. Even worse from the viewpoint of differential lift equations, if N and P are differentiable manifolds modeled on Banach spaces and if φ: $N \to P$ *is* an embedding, there is no general reason to suppose the tangent map $T(\varphi)$: $T(N) \to T(P)$ is again an embedding. The latter phenomenon makes differentiable "pullbacks" of curve lifts and vector fields problematic. The useful notion of embedding in the Banach framework appears to be the following:

Definition. Let φ: $N \to P$ be differentiable. Call φ a *strong embedding* if, for each $x \in N$, there is an open U about $\varphi(x)$ and a differentiable τ: $U \to N$ such that $\tau \circ \varphi|_{\varphi^{-1}(U)} = id|_{\varphi^{-1}(U)}$.

One can easily show (we will not prove) the following:

Facts: (a) If φ is a strong embedding, then so is $T(\varphi)$.

(b) A composition of strong embeddings is a strong embedding.

(c) A strong embedding is an embedding.

(d) If φ: $N \to P$ is an embedding with N finite-dimensional, then φ is a strong embedding.

The full tangential context $(T^k(M), \pi_k)_{k \geq 0}$ is well known, namely, $T^0(M) = M$, and π_k: $T^{k+1}(M) = T(T^k(M)) \to T^k(M)$ is the standard tangent bundle projection. However, it is useful to define the following sequences of sets $({}_kM')_{k\geq 0}$ and $({}_kM)_{k\geq 0}$: ${}_0M' = T(M)$, and

$$(k \geq 1) \qquad {}_kM' = \{x \in T^{k+1}(M):$$
$$0 \leq j \leq k-1 \text{ implies } T^{k-j}(\pi_j)(x) = T(\pi_{k-1})(x)\}$$

Then set ${}_0M = M$, set ${}_1M = T(M)$, and

$$(k \geq 1) \qquad {}_{k+1}M = \{x \in {}_kM': T(\pi_{k-1})(x) = \pi_k(x)\}$$

One has the following result.

Proposition 1 ($k \geq 0$). (1) π_k carries ${}_kM'$ into ${}_kM$.

(2) If $\varphi: N \to T^k(M)$ is differentiable and lies entirely in ${}_kM$, then $T(\varphi)$: $T(N) \to T(T^k(M)) = T^{k+1}(M)$ lies entirely in ${}_kM'$.

(3) For each $x \in {}_kM'$, there is a differentiable K-curve g in $T^k(M)$ lying entirely in ${}_kM$ with $g'(0) = x$.

Proof. We need only consider $k \geq 1$.

(1) Let $i, j = 0, \ldots, k-1$. One has $\pi_k \circ T^{k-i}(\pi_i) = T^{k-i-1}(\pi_i) \circ \pi_k$ and $\pi_k \circ T^{k-j}(\pi_j) = T^{k-j-1}(\pi_j) \circ \pi_k$. Thus, $T^{k-i}(\pi_i)(x) = T^{k-j}(\pi_j)(x)$ implies $T^{k-i-1}(\pi_i)(\pi_k(x)) = T^{k-j-1}(\pi_j)(\pi_k(x))$. For this reason, if $x \in {}_kM'$, then $\pi_k(x)$ is $\in {}_kM$.

(2) Again consider $i, j = 0, \ldots, k-1$. Since φ lies entirely in ${}_kM$, $T^{k-i-1}(\pi_i) \circ \varphi = T^{k-j-1}(\pi_j) \circ \varphi$. Applying T to the latter equation, one has $T^{k-i}(\pi_i) \circ T(\varphi) = T^{k-j}(\pi_j) \circ T(\varphi)$. It follows that $T(\varphi)$ lies entirely in ${}_kM$.

(3) Let M be modeled on the Banach space B. Now for each $j \geq 0$, we can view $T^{j+1}(B)$ as a direct product of Banach spaces: $T^{j+1}(B) = T^j(B) \times T^j(B)$. Under this realization the tangent map τ_j: $T^{j+1}(B) = T^j(B) \times T^j(B) \to T^j(B)$ can be taken to be the direct product projection on the first (leftmost) factor.

Now each τ_j is continuous and linear, whence, for each $m \geq 1$, $T^m(\tau_j)$ is also continuous and linear. Indeed,

$$(*) \quad T^m(\tau_j)(u; v) = (T^{m-1}(\tau_j)(u); T^{m-1}(\tau_j)(v))$$

From this it follows that both ${}_kB$ and ${}_kB'$ are themselves Banach spaces, being coincidence sets for families of continuous linear transformations. In particular, from (*), it follows that ${}_kB' = {}_kB \times {}_kB$.

Let $\theta: X \to Y$ be a local coordinate chart for M, with X open in M and Y open in B. Then θ gives rise, for each $m \geq 1$, to coordinate charts

$$T^m(\theta): \quad (\pi_0 \circ \cdots \circ \pi_{m-1})^{-1}(X) \to (\tau_0 \circ \cdots \circ \tau_{m-1})^{-1}(Y)$$

and

$$T^{m+1}(\theta): \quad (\pi_0 \circ \cdots \circ \pi_m)^{-1}(X) \to (\tau_0 \circ \cdots \circ \tau_m)^{-1}(Y)$$

for $T^m(M)$ and $T^{m+1}(M)$, respectively. For each $i = 0, \ldots, m$,

$$\begin{aligned} &T^{m-i}(\tau_i) \circ T^{m+1}(\theta) \\ &= T^{m-i}(\tau_i) \circ T^{m-i}(T^{i+1}(\theta)) \\ &= T^{m-i}(\tau_i \circ T^{i+1}(\theta)) \\ &= T^{m-i}(T^i(\theta) \circ \pi_i) = T^m(\theta) \circ T^{m-i}(\pi_i) \end{aligned}$$

Because the first term in the preceding equality equals the final term, it follows that $T^{m+1}(\theta)$ carries ${}_mM' \cap (\pi_0 \circ \cdots \circ \pi_m)^{-1}(X)$ onto ${}_mB' \cap (\tau_0 \circ \cdots \circ \tau_m)^{-1}(Y)$ and vice versa, and also $T^{m+1}(\theta)$ carries ${}_{m+1}M \cap (\pi_0 \circ \cdots \circ \pi_m)^{-1}(X)$ onto ${}_{m+1}B \cap (\tau_0 \circ \cdots \circ \tau_m)^{-1}(Y)$ and vice versa.

From the foregoing observation, it follows for our fixed $k \geq 1$ that $T^{k+1}(\theta)$ carries ${}_kM' \cap (\pi_0 \circ \cdots \circ \pi_k)^{-1}(X)$ onto ${}_kB' \cap (\tau_0 \circ \cdots \circ \tau_k)^{-1}(Y)$ and vice versa, and also $T^k(\theta)$ carries ${}_kM \cap (\pi_0 \circ \cdots \circ \pi_{k-1})^{-1}(X)$ onto ${}_kB \cap (\tau_0 \circ \cdots \circ \tau_{k-1})^{-1}(Y)$ and vice versa.

Now, with $x \in {}_kM' \cap (\pi_0 \circ \cdots \circ \pi_k)^{-1}(X)$, let

$$\begin{aligned} T^{k+1}(\theta)(x) &= (u; v) \in {}_kB' \cap (\tau_0 \circ \cdots \circ \tau_k)^{-1}(Y) \\ &= ({}_kB \times {}_kB) \cap (\tau_0 \circ \cdots \circ \tau_k)^{-1}(Y) \end{aligned}$$

Let $h = h(t)$ be the K-curve in ${}_kB$ given by $h(t) = tv + u$. Clearly, $h'(0) = (h(0); d/dt[h(t)]|_{t=0}) = (u; v)$. Moreover, if t is taken sufficiently close to 0, $h(t)$ is always in $(\tau_0 \circ \cdots \circ \tau_{k-1})^{-1}(Y)$, which is open about u. Thus, we have produced a differentiable K-curve h lying entirely in ${}_kB \cap (\tau_0 \circ \cdots \circ \tau_{k-1})^{-1}(Y)$ with $h'(0) = (u; v)$. Then $g = T^k(\theta)^{-1} \circ h$ meets the requirements of the proposition.

Remark. It is thus clear that, even though ${}_kM'$ and ${}_kM$ are just defined as sets, with no topological or differentiable structure, ${}_kM'$ *behaves* as though it were the tangent bundle manifold over ${}_kM$, with the restriction of π_k as tangent bundle projection. This is the technical key to the entire paper. In particular, part (3) of the proposition is the reason, as will be seen, we can assert that the Bowman tangential context captures *every* higher order differential lift equation from the full tangential context.

All we shall need from the Bowman context $({}^kM, \pi^k, {}_kI, {}_k\pi)_{k\geq 0}$ are the following global details:

(a) Each ${}^kM$ is a differentiable manifold, with ${}^0M = M$ and ${}^1M = T(M)$.

(b) π^k: $T({}^kM) \to {}^kM$ denotes the tangent bundle projection.

(c) ${}_kI$: ${}^{k+1}M \to T({}^kM)$ is a strong embedding such that ${}_0I = id_{T(M)}$, and

$$(k \geq 1) \qquad {}_kI({}^{k+1}M) = \{x \in T({}^kM): ({}_{k-1}I \circ \pi^k)(x) = T(\pi^{k-1} \circ {}_{k-1}I)(x)\}$$

(d) ${}_k\pi = \pi^k \circ {}_kI$: ${}^{k+1}M \to {}^kM$.

Let N be a differentiable manifold with Bowman resolution $({}^kN, \mu^k, {}_kJ, {}_k\mu)_{k\geq 0}$, and let F: $M \to N$ be any differentiable map. Based solely on (a)–(d) it can be shown inductively that there is a sequence of differentiable maps ${}^kF$: ${}^kM \to {}^kN$ unique with the properties that ${}^0F = F$, and ${}_kJ \circ {}^{k+1}F = T({}^kF) \circ {}_kI$. Moreover, as a direct calculation on these defining relations shows, ${}_k\mu \circ {}^{k+1}F = {}^kF \circ {}_k\pi$.

Let U be a nonempty open set in K, and let W: $U \to T(U) = U \times K$ be the *standard vector field* given by $W(t) = (t; 1)$. [Then, for any differentiable K-curve g, the differential lift of g is the K-curve $g' = T(g) \circ W$.] Based solely on (a)–(d) it can be shown inductively that, given a differentiable K-curve f: $U \to M$, there is a sequence of differentiable K-curves $f^{(k)}$: $U \to {}^kM$ unique with the properties that $f^{(0)} = f$, and ${}_kI \circ f^{(k+1)} = T(f^{(k)}) \circ W$. Moreover, as a direct calculation on these defining relations shows, ${}_k\pi \circ f^{(k+1)} = f^{(k)}$.

Definitions. Let $\overline{g}$: $U \to {}^kM$ be a differentiable K-curve. Call $\overline{g}$ *k-suitable* if $\overline{g}$ has the form $\overline{g} = f^{(k)}$, where f is a K-curve in M.

Let $\overline{Q}$ be a subset/submanifold in ${}^kM$, and let $\overline{V}$: $\overline{Q} \to T(\overline{Q})$ be a differentiable vector field. Call $\overline{V}$ *k-suitable* if $\overline{V}$ has the property that each $\overline{g}$ in $\overline{Q}$ satisfying $\overline{g} = V \circ \overline{g}$ is k-suitable.

Proposition 2. Let $k \geq 1$, and let $\overline{g}$: $U \to {}^kM$ be a differentiable K-curve. Then $\overline{g}$ is k-suitable if and only if $T({}_{k-1}\pi) \circ \overline{g}' = {}_{k-1}I \circ \overline{g}$ if and only if $\overline{g}'$ lies entirely in ${}_kI({}^{k+1}M)$. In this event, $f = ({}_0\pi \circ \cdots \circ {}_{k-1}\pi \circ g)$.

Proof. Let τ: $T(U) = U \times K \to U$ denote the tangent bundle projection: $\tau(t; s) = t$. Now

$$\begin{aligned} {}_{k-1}I \circ \overline{g} &= {}_{k-1}I \circ \overline{g} \circ (\tau \circ W) = {}_{k-1}I \circ (\overline{g} \circ \tau) \circ W \\ &= {}_{k-1}I \circ (\pi^k \circ T(\overline{g})) \circ W = ({}_{k-1}I \circ \pi^k) \circ (T(\overline{g}) \circ W) \end{aligned}$$

So the second "if and only if" assertion follows from the first by characterization (c) of the embeddings ${}_kI$.

To see that the first assertion holds, assume, first of all, that $\overline{g} = f^{(k)}$. Then

$$\begin{aligned} T({}_{k-1}\pi) \circ \overline{g}' &= T({}_{k-1}\pi) \circ T(\overline{g}) \circ W \\ &= T({}_{k-1}\pi \circ f^{(k)}) \circ W = T(f^{(k-1)}) \circ W = {}_{k-1}I \circ f^{(k)} = {}_{k-1}I \circ \overline{g} \end{aligned}$$

as asserted.

To see the converse (including the final form for f), we argue by induction on k. For $k = 1$, the assumption on $\overline{g}$ is that $T({}_0\pi) \circ T(\overline{g}) \circ W = {}_0I \circ \overline{g} = \overline{g}$. That is, $T({}_0\pi \circ \overline{g}) \circ W = \overline{g}$, which says precisely that $({}_0\pi \circ \overline{g})' = \overline{g}$, as asserted.

Assume, inductively, that the result holds for k. Consider $\bar{g}: U \to {}^{k+1}M$ such that $T({}_k\pi) \circ T(\bar{g}) \circ W = {}_kI \circ \bar{g}$. Then consider the projected curve $[{}_k\pi \circ g]: U \to {}^kM$. One has

$$\begin{aligned} &T({}_{k-1}\pi) \circ T([{}_k\pi \circ \bar{g}]) \circ W \\ &\quad = T({}_{k-1}\pi) \circ [T({}_k\pi) \circ T(g) \circ W] \\ &\quad = T({}_{k-1}\pi) \circ [{}_kI \circ \bar{g}] = [T({}_{k-1}\pi) \circ {}_kI] \circ \bar{g} \\ &\quad = {}_{k-1}I \circ [{}_k\pi \circ \bar{g}] \end{aligned}$$

whence, by the inductive assumption, $[{}_k\pi \circ \bar{g}] = ({}_0\pi \circ \cdots \circ {}_{k-1}\pi \circ [{}_k\pi \circ g])^{(k)}$. But

$${}_kI \circ \bar{g} = T({}_k\pi) \circ T(\bar{g}) \circ W = T([{}_k\pi \circ \bar{g}]) \circ W$$

which

$$\begin{aligned} &= T(({}_0\pi \circ \cdots \circ {}_{k-1}\pi \circ {}_k\pi \circ \bar{g})^{(k)}) \circ W \\ &= {}_kI \circ ({}_0\pi \circ \cdots \circ {}_k\pi \circ \bar{g})^{(k+1)} \end{aligned}$$

Thus, since ${}_kI$ is one to one, $\bar{g} = ({}_0\pi \circ \cdots \circ {}_k\pi \circ \bar{g})^{(k+1)}$, which completes the inductive step and the proof of the proposition.

Corollary ($k \geq 1$). Let $\bar{Q}$ be a subset/submanifold in ${}^kM$, and regard $T(\bar{Q})$ as a subset in $T({}^kM)$. Let $\bar{V}: \bar{Q} \to T(\bar{Q})$ be a differentiable vector field. Then $\bar{V}$ is k-suitable if and only if $T({}_{k-1}\pi) \circ \bar{V} = {}_{k-1}I$ on $\bar{Q}$ if and only if $V(\bar{Q}) \subseteq {}_kI({}^{k+1}M)$.

Proof. The second "if and only if" assertion follows from the first, since

$$({}_{k-1}I \circ \pi^k) \circ \bar{V} = {}_{k-1}I \circ (\pi^k \circ \bar{V}) = {}_{k-1}I \circ \mathrm{id}_{\bar{Q}} = {}_{k-1}I \quad \text{on} \quad \bar{Q}$$

As to the first assertion, assume, first of all, that $T({}_{k-1}\pi) \circ \bar{V} = {}_{k-1}I$ on $\bar{Q}$. Let $\bar{g}$ be any differentiable K-curve in $\bar{Q}$ such that $\bar{g}' = \bar{V} \circ \bar{g}$. Then

$$T({}_{k-1}\pi) \circ \bar{g}' = T({}_{k-1}\pi) \circ T(\bar{g}) \circ W = T({}_{k-1}\pi) \circ \bar{V} \circ \bar{g} = {}_{k-1}I \circ \bar{g}$$

whence $\bar{g}$ is k-suitable by the proposition.

Conversely, assume $\bar{V}$ is k-suitable, and consider any $\bar{q} \in \bar{Q}$. By the fundamental existence/uniqueness theorem (smoothness of $\bar{V}$ near $\bar{q}$ being the only issue) there is a differentiable K-curve $\bar{g}$ in $\bar{Q}$ such that $\bar{g}' = \bar{V} \circ \bar{g}$ with $\bar{g}(0) = \bar{q}$. Because $\bar{g}$ is k-suitable by assumption, one has

$$\begin{aligned} T({}_{k-1}\pi)(\bar{V}(\bar{q})) &= T({}_{k-1}\pi)(\bar{V}(\bar{g}(0))) \\ &= T({}_{k-1}\pi)(\bar{g}'(0)) = {}_{k-1}I(\bar{g}(0)) = {}_{k-1}I(\bar{q}) \end{aligned}$$

Thus, $T({}_{k-1}\pi) \circ \bar{V} = {}_{k-1}I$ on $\bar{Q}$, as required.

We will return to the issue of k-suitability of curves and vector fields after establishing the embedding linkage between the two tangential contexts. For now, observe that the k-suitability criterion $T({}_{k-1}\pi) \circ \overline{V} = {}_{k-1}I$ generalizes the ($k = 1$) condition encountered in the closely related contexts of sprays and of Hamiltonian vector fields associated with regular Lagrangians. [See Ambrose *et al.* (1960) for the classic development of the former idea, and Abraham (1967) for a development of the latter idea.]

Theorem 1. Let $({}^kM, \pi^k, {}_kI, {}_k\pi)_{k\geq 0}$ be the Bowman resolution for the differentiable manifold M. Let $\varphi_0 = id_M$: ${}^0M \to T^0(M)$ and, inductively, let $\varphi_{k+1} = T(\varphi_k) \circ {}_kI$: ${}^{k+1}M \to T(T^k(M)) = T^{k+1}(M)$.

(1) Each φ_k is a strong embedding into/onto the set ${}_kM$, and each $T(\varphi_k)$ is a strong embedding into/onto the set ${}_kM'$.

(2) For each k, $\varphi_k \circ {}_k\pi = \pi_k \circ \varphi_{k+1}$.

(3) Let N be a differentiable manifold with Bowman resolution $({}^kN, \mu^k, {}_kJ, {}_k\mu)_{k\geq 0}$, and let F: $M \to N$ be differentiable. Let $(\psi_k)_{k\geq 0}$ be the sequence of embeddings over N corresponding to the sequence of embeddings $(\varphi_k)_{k\geq 0}$ over M. Then, for each k, $\psi_k \circ {}^kF = T^k(F) \circ \varphi_k$.

Proof. For all three assertions, only the cases $k \geq 1$ need argument. For the first and third assertions, the argument is inductive.

(1) $\varphi_1 = T(\varphi_0) \circ {}_0I = T(\mathrm{id}_M) \circ \mathrm{id}_{T(M)}$ is simply an identity map, whence $T(\varphi_1)$ is as well. Assume, inductively, that result (1) holds in its entirety for k, and consider circumstances for $k + 1$. To begin with, $\varphi_{k+1} = T(\varphi)_k \circ {}_kI$ is again a strong embedding by the preliminary facts on strong embeddings. Then, successively, $T(\varphi_{k+1})$ is also a strong embedding.

Since φ_k is into ${}_kM$, part (2) of Proposition 1 ensures that $T(\varphi_k)$ is into ${}_kM'$. Thus, $\varphi_{k+1} = T(\varphi_k) \circ {}_kI$ is also into ${}_kM'$. In fact, as we shall see, φ_{k+1} is into ${}_{k+1}M \subseteq {}_kM'$, i.e., $T(\pi_{k-1}) \circ \varphi_{k+1} = \pi_k \circ \varphi_{k+1}$. For one has

$$
\begin{aligned}
&T(\pi_{k-1}) \circ \varphi_{k+1} \\
&\quad= T(\pi_{k-1}) \circ [T(\varphi_k) \circ {}_kI] = [T(\pi_{k-1}) \circ T(\varphi_k)] \circ {}_kI \\
&\quad= T(\pi_{k-1} \circ \varphi_k) \circ {}_kI = T(\pi_{k-1} \circ [T(\varphi_{k-1}) \circ {}_{k-1}I]) \circ {}_kI \\
&\quad= T([\pi_{k-1} \circ T(\varphi_{k-1})] \circ {}_{k-1}I) \circ {}_kI = T([\varphi_{k-1} \circ \pi^{k-1}] \circ {}_{k-1}I) \circ {}_kI \\
&\quad= T(\varphi_{k-1} \circ [\pi^{k-1} \circ {}_{k-1}I]) \circ {}_kI = T(\varphi_{k-1}) \circ (T[\pi^{k-1} \circ {}_{k-1}I] \circ {}_kI) \\
&\quad= T(\varphi_{k-1}) \circ ([{}_{k-1}I \circ \pi^k] \circ {}_kI) = (T(\varphi_{k-1}) \circ {}_{k-1}I) \circ (\pi^k \circ {}_kI) \\
&\quad= \varphi_k \circ (\pi^k \circ {}_kI) = (\varphi_k \circ \pi^k) \circ {}_kI = (\pi_k \circ T(\varphi_k)) \circ {}_kI \\
&\quad= \pi_k \circ [T(\varphi_k) \circ {}_kI] = \pi_k \circ \varphi_{k+1}
\end{aligned}
$$

as required.

To see that φ_{k+1} is *onto* ${}_{k+1}M$, consider any $x \in {}_{k+1}M$. Because $x \in {}_kM'$ as well, part (3) of Proposition 1 ensures that there is a differentiable K-curve g in $T^k(M)$ lying entirely in ${}_kM$ with $g'(0) = x$. Then the embedding φ_k, which is onto ${}_kM$ by the inductive hypothesis, pulls g back (uniquely) to a differentiable K-curve $\bar{g}$ in ${}^kM$ such that $g = \varphi_k \circ \bar{g}$.

Now $x = g'(0) = [\varphi_k \circ \bar{g}]'(0) = T(\varphi_k)(\bar{g}'(0))$. We will show that $\bar{g}'(0) \in {}_kI({}^{k+1}M)$, whence, with $\bar{g}'(0) = {}_kI(y)$, it follows that $x = (T(\varphi_k) \circ {}_kI)(y) = \varphi_{k+1}(y)$. Thus, we must show that $T({}_{k-1}\pi)(\bar{g}'(0)) = {}_{k-1}I(\bar{g}(0))$. One has

$$
\begin{aligned}
&T(\varphi_{k-1})(T({}_{k-1}\pi)(\bar{g}'(0)))\\
&\quad = T(\varphi_{k-1} \circ [{}_{k-1}\pi] \circ \bar{g})(W(0))\\
&\quad = T(\varphi_{k-1} \circ [\pi^{k-1} \circ {}_{k-1}I] \circ \bar{g})(W(0))\\
&\quad = T([\varphi_{k-1} \circ \pi^{k-1}] \circ {}_{k-1}I \circ \bar{g})(W(0))\\
&\quad = T([\pi_{k-1} \circ T(\varphi_{k-1})] \circ {}_{k-1}I \circ \bar{g})(W(0))\\
&\quad = T(\pi_{k-1} \circ [T(\varphi_{k-1}) \circ {}_{k-1}I] \circ \bar{g})(W(0)) = T(\pi_{k-1} \circ \varphi_k \circ \bar{g})(W(0))\\
&\quad = T(\pi_{k-1} \circ g)(W(0)) = T(\pi_{k-1})[(T(g) \circ W)(0)] = T(\pi_{k-1})(g'(0))\\
&\quad = T(\pi_{k-1})(x)
\end{aligned}
$$

But we assume $x \in {}_{k+1}M$. Hence,

$$
\begin{aligned}
T(\pi_{k-1})(x) &= \pi_k(x) = \pi_k(g'(0)) = g(0) = \varphi_k(\bar{g}(0))\\
&= [T(\varphi_{k-1}) \circ {}_{k-1}I](\bar{g}(0)) = T(\varphi_{k-1})([{}_{k-1}I \circ \bar{g}](0))
\end{aligned}
$$

In summary,

$$
T(\varphi_{k-1})(T({}_{k-1}\pi)(\bar{g}'(0))) = T(\varphi_{k-1})([{}_{k-1}I \circ \bar{g}](0))
$$

Thus, since $T(\varphi_{k-1})$ is one to one, it follows that $T({}_{k-1}\pi)(\bar{g}'(0)) = {}_{k-1}I(\bar{g}(0))$, as required. Hence, φ_{k+1} is indeed onto ${}_{k+1}M$.

Finally, from part (2) of Proposition 1, we know that $T(\varphi_{k+1})$ is into ${}_{k+1}M'$, since φ_{k+1} is into ${}_{k+1}M$. To see that $T(\varphi_{k+1})$ is *onto* ${}_{k+1}M'$, let $u \in {}_{k+1}M'$. Then, per part (3) of Proposition 1, let h be a differentiable K-curve in $T^{k+1}(M)$ lying entirely in ${}_{k+1}M$ with $h'(0) = u$. Then, because φ_{k+1} is an embedding and onto ${}_{k+1}M$, h pulls back (uniquely) to a differentiable K-curve $\bar{h}$ in ${}^{k+1}M$ such that $h = \varphi_{k+1} \circ \bar{h}$. Then

$$
T(\varphi_{k+1})(\bar{h}'(0)) = (\varphi_{k+1} \circ \bar{h})'(0) = h'(0) = u
$$

Hence, $T(\varphi_{k+1})$ is indeed onto ${}_{k+1}M$. This completes the inductive step and thereby the proof of result (1).

(2) One has

$$\varphi_k \circ {}_k\pi = \varphi_k \circ [\pi^k \circ {}_kI] = [\varphi_k \circ \pi^k] \circ {}_kI = [\pi_k \circ T(\varphi_k)] \circ {}_kI$$
$$= \pi_k \circ [T(\varphi_k) \circ {}_kI] = \pi_k \circ \varphi_{k+1}$$

(3) For $k = 1$, the result is a tautology:

$$\psi_1 \circ {}^1F = \mathrm{id}_{T(N)} \circ T(F) = T(F) \circ \mathrm{id}_{T(M)} = T(F) \circ \varphi_1$$

Assume, inductively, that the result holds for k, i.e., that $\psi_k \circ {}^kF = T^k(F) \circ \varphi_k$. Then apply T to the latter equality to obtain $T(\psi_k) \circ T({}^kF) = T^{k+1}(F) \circ T(\varphi_k)$. To this equality apply ${}_kI$ first to obtain

$$T(\psi_k) \circ T({}^kF) \circ {}_kI = T^{k+1}(F) \circ T(\varphi_k) \circ {}_kI = T^{k+1}(F) \circ \varphi_{k+1}$$

On the other hand,

$$T(\psi_k) \circ [T({}^kF) \circ {}_kI] = T(\psi_k) \circ [{}_kJ \circ {}^{k+1}F] = [T(\psi_k) \circ {}_kJ] \circ {}^{k+1}F$$
$$= \psi_{k+1} \circ {}^{k+1}F$$

Thus, $\psi_{k+1} \circ {}^{k+1}F = T^{k+1}(F) \circ \varphi_{k+1}$, which completes the inductive step and, thereby, the proof of result (3) and the theorem as a whole.

Remark. Although there is no pressing need to do so, by means of the strong embeddings φ_k and $T(\varphi_k)$ one can transfer the tangent bundle structure π^k: $T({}^kM) \to {}^kM$ to π_k: ${}_kM' \to {}_kM$ and regard the latter as a strongly embedded linear subbundle in π_k: $T^{k+1}(M) = T(T^k(M)) \to T^k(M)$. As the following section will show, however, we can safely abandon the $T^k(\cdot)$ context: it is entirely irrelevant to the study of higher order differential lift equations.

TRANSFER OF k-SUITABILITY OF CURVES AND VECTOR FIELDS

In the *full* tangential framework over a manifold M, the notion of successive differential curve lifts is as follows: If f: $U \to M$ is a differentiable K-curve, then $f^{[0]} = f$: $U \to T^0(M) = M$ and, inductively, $f^{[k+1]}$: $U \to T^{k+1}(M)$ is given by $f^{[k+1]}(t) = (f^{[k]})'(t) = T(f^{[k]})(t; 1)$.

Corresponding to the notion of k-suitability in the Bowman tangential context (as developed in the preceding section), call a differentiable K-curve g in $T^k(M)$ *k-suitable* if g has the form $g = f^{[k]}$, where f is a K-curve in M. Similarly, if Q is a subset/submanifold in $T^k(M)$ and V: $Q \to T(Q)$ is a differentiable vector field, call V *k-suitable* if every differentiable K-curve g satisfying $g' = V \circ g$ is k-suitable.

In Pond (1997) it is shown (Theorem 1 and its corollary, essentially) that:

(i) Any g in $T^k(M)$ is k-suitable if and only if g lies entirely in ${}_kM$ and g' lies entirely in ${}_{k+1}M$.

(ii) V is k-suitable if and only if $Q \subseteq {}_kM$ and $V(Q) \subseteq {}_{k+1}M$.

Putting the foregoing results together with Proposition 2, its corollary, and Theorem 1 in the present paper, we obtain:

Theorem 2. (1) A differentiable K-curve g in $T^k(M)$ is k-suitable if and only if $g = \varphi_k \circ \bar{g}$, where $\bar{g}$ is a differentiable k-suitable K-curve in ${}^kM$.

(2) Let Q be a subset/submanifold in $T^k(M)$. Let V: $Q \to T(Q)$ be a differentiable vector field. Then V is k-suitable if and only if $Q = \varphi_k(\overline{Q})$ and $V = T(\varphi_k) \circ \overline{V} \circ \varphi_k^{-1}$, where $\overline{Q}$ is a subset/submanifold in ${}^kM$ and $\overline{V}$: $\overline{Q} \to T(\overline{Q})$ is a k-suitable differentiable vector field.

Proof ($k \geq 1$) (1) Assume that g is k-suitable. Then, by the result cited above, g lies entirely in ${}_kM$ and $T(\pi_{k-1}) \circ g' = g$. Let $\bar{g}$ be the differentiable pullback of g to ${}^kM$ by φ_k, i.e., $g = \varphi_k \circ \bar{g}$. Then one has

$$\begin{aligned}
&T(\varphi_{k-1}) \circ [{}_{k-1}I \circ \bar{g}] \\
&\quad = [T(\varphi_{k-1}) \circ {}_{k-1}I] \circ \bar{g} = \varphi_k \circ \bar{g} = g \\
&\quad = T(\pi_{k-1}) \circ g' = T(\pi_{k-1} \circ g) \circ W = T(\pi_{k-1} \circ [\varphi_k \circ \bar{g}]) \circ W \\
&\quad = T([\pi_{k-1} \circ \varphi_k] \circ \bar{g}) \circ W = T(\pi_{k-1} \circ \varphi_k) \circ \bar{g}' = T(\varphi_{k-1} \circ {}_{k-1}\pi) \circ \bar{g}' \\
&\quad = T(\varphi_{k-1}) \circ [T({}_{k-1}\pi) \circ \bar{g}']
\end{aligned}$$

Since $T(\varphi_{k-1})$ is one to one, $[{}_{k-1}I \circ \bar{g}] = [T({}_{k-1}\pi) \circ \bar{g}']$, whence $\bar{g}$ is k-suitable by the criterion in Proposition 2.

On the other hand, let $\bar{g}$ be k-suitable in ${}^kM$, and consider $g = \varphi_k \circ \bar{g}$. One has

$$\begin{aligned}
&T(\pi_{k-1}) \circ g' \\
&\quad = T(\pi_{k-1}) \circ [T(\varphi_k) \circ \bar{g}'] \\
&\quad = [T(\pi_{k-1}) \circ T(\varphi_k)] \circ T(\bar{g}) \circ W = T([\pi_{k-1} \circ \varphi_k] \circ \bar{g}) \circ W \\
&\quad = T([\varphi_{k-1} \circ {}_{k-1}\pi] \circ \bar{g}) \circ W = T(\varphi_{k-1}) \circ [T({}_{k-1}\pi) \circ \bar{g}'] \\
&\quad = T(\varphi_{k-1}) \circ [{}_{k-1}I \circ \bar{g}] = [T(\varphi_{k-1}) \circ {}_{k-1}I] \circ \bar{g} = \varphi_k \circ \bar{g} = g
\end{aligned}$$

Thus, g satisfies the criterion cited above for k-suitability in the full tangential context.

(2) Assume V is k-suitable in the full tangential context. Then, by the criterion cited above, $Q \subseteq {}_kM$. Let $\overline{Q}$ be the subset/submanifold in ${}^kM$ such that $\varphi_k(\overline{Q}) = Q$. Let $\overline{V}$: $\overline{Q} \to T(\overline{Q})$ be the differentiable vector field given by $\overline{V} = T(\varphi_k)^{-1} \circ V \circ \varphi_k$. Clearly K-curves $\bar{g}$ satisfying $\bar{g}' = \overline{V} \circ \bar{g}$ are related to K-curves g satisfying $g' = V \circ g$ by $g = \varphi_k \circ \bar{g}$. Thus, $\overline{V}$ is k-suitable by result (1).

On the other hand, let $\overline{V}$: $\overline{Q} \to T(\overline{Q})$ be differentiable and k-suitable, where $\overline{Q}$ is a subset/submanifold in ${}^kM$. Let $Q = \varphi_k(\overline{Q})$, and let V: $Q \to T(Q)$ be given by $V = T(\varphi_k) \circ \overline{V} \circ \varphi_k^{-1}$. Again, K-curves g satisfying $g' = V \circ g$ are related to K-curves $\overline{g}$ satisfying $\overline{g}' = \overline{V} \circ \overline{g}$ by $g = \varphi_k \circ \overline{g}$. Thus, V is k-suitable in the full tangential context by result (1).

Remark. It is to be emphasized that the Bowman tangential context captures *all* k-suitable curves and *all* k-suitable vector fields from the full tangential context.

Example 1. M is an open set in a Banach space B. This seemingly homely generic example is presented for two reasons: (a) To concretize the abstract development in the main body of the paper, and (b) to make the case that, aside from sheer mathematics, the restricted tangential context is "right" from the viewpoint of physical problems, whereas the full tangential context is not.

Following the usual convention $T(M) = M \times B$ to higher differential levels, one has $T^{k+1}(M) = T^k(M) \times T^k(B)$ for each k. Thus, the number of coordinate positions describing elements *doubles* as k increases stepwise—a circumstance hardly conducive to a succession of interpretations: instantaneous position, position/velocity, position/velocity/acceleration, etc. The latter progression requires a simple *arithmetic* increase in degrees of freedom as k increases.

On the other hand, the *restricted* tangential resolution for M is as follows:

(a) ${}^0M = M$, while ${}^{k+1}M = M \times B^{k+1}$. Thus, the progression ${}^0M$, ${}^1M$, ${}^2M, \ldots$ *does* admit successive interpretations: instantaneous position, position/velocity, position/velocity/acceleration, etc.

(b) The tangential projection π^k: $T({}^kM) = {}^kM \times B^{k+1} \to {}^kM$ is, of course, just the direct product projection on the first (leftmost) factor.

(c) ${}_kI$: ${}^{k+1}M \to T({}^kM)$ is given by

$${}_kI(x_0, \ldots, x_{k+1}) = (x_0, \ldots, x_k; x_1, \ldots, x_{k+1})$$

(Note: ${}_kI$ is a *strong* embedding because it has a differentiable left inverse λ_k: $T({}^kM) \to {}^{k+1}M$ given by $\lambda_k(x_0, \ldots, x_k; y_0, \ldots, y_k) = (x_0, \ldots, x_k, y_k)$; the local version of this in the general manifold setting is the reason those embeddings are always strong.)

(d) ${}_k\pi = \pi^k \circ {}_kI$: ${}^{k+1}M \to {}^kM$ is given by

$${}_k\pi(x_0, \ldots, x_{k+1}) = (x_0, \ldots, x_k)$$

If N is an open set in a Banach space C and if F: $M \to N$ is differentiable, then the sequence of differentiable maps ${}^kF$: ${}^kM \to {}^kN$ develops inductively as follows: Define a sequence of (differentiable) maps $F_j: {}^jM \to C$ by $F_0(x_0) = F(x_0)$, and

$$F_{j+1}(x_0, \ldots, x_{j+1}) = D(F_j)(x_0, \ldots, x_j)(x_1, \ldots, x_{j+1})$$

where D denotes the total differential operator for a function. Then ${}^kF(x_0, \ldots, x_k) = (F_0(x_0), \ldots, F_k(x_0, \ldots, x_k))$.

If f is a differentiable K-curve in M, then ($k \geq 1$)

$$f^{(k)}(t) = (f(t), d/dt[f(t)], \ldots, d^k/dt^k[f(t)])$$

Thus, a k-suitable $\overline{g} = f^{(k)}$ always represents a coherent progression of motion states under the physical interpretation above, whereas a general $\overline{g}$ does not.

A general differentiable vector field $\overline{V}$ over ${}^kM$ takes the form

$$\begin{aligned} &\overline{V}(x_0, \ldots, x_k) \\ &\quad = (x_0, \ldots, x_k; G_0(x_0, \ldots, x_k), \ldots, G_k(x_0, \ldots, x_k)) \end{aligned}$$

where each G_i is a differentiable function from ${}^kM$ to B. With $k \geq 1$, V is k-suitable if and only if $G_i(x_0, \ldots, x_k) = x_{i+1}$ for each $i = 0, \ldots, k - 1$. Thus, for a k-suitable $\overline{V}$, $\overline{g} = f^{(k)}$ satisfies $\overline{g}' = V \circ \overline{g}$ if and only if

$$d^{k+1}/dt^{k+1}[f(t)] = G_k(f(t), \ldots, d^k/dt^k[f(t)])$$

which agrees with the classical formulation.

Finally, to describe the strong embeddings φ_k and $T(\varphi_k)$ for $k \geq 1$, we adopt an numerative, binary indexing scheme for elements in the higher differential level full tangential extensions $T^m(M)$. Thus, $(x_0; x_1) \in T(M)$, $(x_{00}, x_{01}; x_{10}, x_{11}) \in T^2(M)$, $(x_{000}, \ldots, x_{111}) \in T^3(M)$, etc.

Now each φ_k: ${}^kM \to T^k(M)$ is the restriction of a continuous linear map. Using this fact, one can show (inductively) that φ_k: ${}^kM \to T^k(M)$ is given by

$$\varphi_k(x_0, \ldots, x_k) = (x_{0\cdots 0}, \ldots, x_{i_1\cdots i_k}, \ldots, x_{1\cdots 1})$$

where $x_{i_1\cdots i_k} = x_j$ if and only if $i_1 + \cdots + i_k = j$. It follows that $(x_{0\cdots 0}, \ldots, x_{i_1\cdots i_k}, \ldots, x_{1\cdots 1}) \in {}_kM$ if and only if $x_{i_1\cdots i_k} = x_{j_1\cdots j_k}$ whenever $i_1 + \cdots + i_k = j_1 + \cdots + j_k$. Similarly, $(x_{0\cdots 0}, \ldots, x_{i_1\cdots i_{k+1}}, \ldots, x_{1\cdots 1}) \in {}_kM'$ if and only if $x_{0i_2\cdots i_{k+1}} = x_{0j_2\cdots j_{k+1}}$ and $x_{1i_2\cdots i_{k+1}} = x_{1j_2\cdots j_{k+1}}$ whenever $i_2 + \cdots + i_{k+1} = j_2 + \cdots + j_{k+1}$.

Example 2. Higher Order Lagrangian Hamiltonian Structures. How one best defines (more importantly, how one *derives*) these objects depends on one's purpose. However, if $k > 1$, there is always a difficulty concerning k-suitability of the Hamiltonian vector field defining the equation of motion.

Namely, regard Q, $T(Q)$, and $T^2(Q)$ as subsets/submanifolds in $T^{k-1}(M)$, $T^k(M)$, and $T^{k+1}(M)$, respectively. Let $V = V_L$: $T(Q) \to T^2(Q)$ be the Hamiltonian vector field associated with a regular Lagrangian L over Q. It is a standard theorem in mechanics (see Abraham, 1967, p. 122, for a proof) that, in our language, V is automatically 1-suitable. Thus, if g: $U \to T(Q)$ is any differentiable curve satisfying $g' = V \circ g$, then g has the form $g = h'$, where h lies in Q. However, there is no assurance that h itself has the form $h = f^{[k-1]}$, where f lies in M. Thus, g does not necessarily represent the kth-order differential lift of a motion curve in the underlying position/configuration manifold M.

Indeed, from the discussion preceding Theorem 2, for V as above to be k-suitable, it is necessary and sufficient that $T(Q) \subseteq {}_kM \subseteq T^k(M)$. Unfortunately, if $k > 1$, one can *never* have *all* of $T(Q)$ realized as a subset/submanifold in ${}_kM$, unless Q is discrete (singletons are open sets).

For suppose $T(Q) \subseteq {}_kM$, and consider (Theorem 1) the subset/submanifold $P = \varphi_k^{-1}(T(Q))$ in ${}^kM$. Let $\overline{Q} = {}_{k-1}\pi(P)$ in ${}^{k-1}M$. One has

$$\varphi_{k-1}(\overline{Q}) = (\varphi_{k-1} \circ {}_{k-1}\pi)(P) = (\pi_{k-1} \circ \varphi_k)(P)$$

by part (2) of Theorem 1. But the latter quantity is equal to $\pi_{k-1}(\varphi_k(\varphi_k^{-1}(T(Q)))) = \pi_{k-1}(T(Q)) = Q$. That is, $\overline{Q} = \varphi_{k-1}^{-1}(Q)$ is necessarily a subset/submanifold in ${}^{k-1}M$. On the other hand, one has

$$T(\overline{Q}) = T(\varphi_{k-1}^{-1})(T(Q)) = T(\varphi_{k-1})^{-1}(T(Q)) = T(\varphi_{k-1})^{-1}(\varphi_k(P))$$
$$= T(\varphi_{k-1})^{-1}([T(\varphi_{k-1}) \circ {}_{k-1}I](P)) = {}_{k-1}I(P) \circ {}_{k-1}I({}^kM)$$

Thus we have both $\overline{Q} \subseteq {}^{k-1}M$ and $T(\overline{Q}) \subseteq {}_{k-1}I({}^kM) \subseteq T({}^{k-1}M)$.

Now let $u \in T(\overline{Q})$. Let θ: $X \to Y$ be a local coordinate chart for M, where X is open in M and where Y is open in the Banach space B. The θ gives rise to coordinate charts:

$${}^{k-1}\theta\colon \quad ({}_0\pi \circ \cdots \circ {}_{k-2}\pi)^{-1}(X) \to Y \times B^{k-1}$$

and

$$T({}^{k-1}\theta)\colon \quad ({}_0\pi \circ \cdots \circ {}_{k-2}\pi \circ \pi^{k-1})^{-1}(X) \to (Y \times B^{k-1}) \times B^k$$

for ${}^{k-1}M$ and $T({}^{k-1}M)$, respectively. We can assume $u \in T(\overline{Q}) \cap ({}_0\pi \circ \cdots \circ {}_{k-2}\pi \circ \pi^{k-1})^{-1}(X)$.

Since, by assumption, $u \in {}_{k-1}I({}^kM)$, it follows that $T({}^{k-1}\theta)(u)$ has the form $(y_0, \ldots, y_{k-1}; y_1, \ldots, y_{k-1}, y_k)$. But, for any nonzero, nonunity real number λ, one must also have $(y_0, \ldots, y_{k-1}; \lambda y_1, \ldots, \lambda y_{k-1}, \lambda y_k)$ in

$$T({}^{k-1}\theta)(T(\overline{Q}) \cap ({}_0\pi \circ \cdots \pi_{k-2} \circ \pi^{k-1})^{-1}(X))$$

Since $T(\overline{Q}) \subseteq {}_{k-1}I({}^kM)$, we are forced to conclude $\lambda y_1 = y_1, \ldots, \lambda y_{k-1} = y_{k-1}$, whence $0 = y_1 = \cdots = y_{k-1}$.

Now let $e = e(t) = (e_0(t), \ldots, e_{k-1}(t))$ be a differentiable curve with $e'(0) = T({}^{k-1}\theta)(u) = (y_0, 0, \ldots, 0; 0, \ldots, 0, y_k)$, and with $e(t) \in {}^{k-1}\theta(\overline{Q})$

for all t sufficiently near 0. The argument given above for the form that $T(^{k-1}\theta)(u)$ must take also applies to any $v \in T(\overline{Q})$ if v is taken sufficiently close to u. It follows that any $z \in \overline{Q}$ sufficiently close to $^{k-1}\theta(u)$ has $^{k-1}\theta(z)$ in the form $(z_0, 0, \ldots, 0)$. In particular, for all t sufficiently close to 0, $e(t) = (e_0(t), 0, \ldots, 0)$. Thus, $d/dt[e_{k-1}(t)]|_{t=0} = y_k = 0$. In summary, u is necessarily a zero tangent vector. The only way this can happen, since u is arbitrary in $T(\overline{Q})$, is for $\overline{Q}$ to be a discrete manifold to begin with, whence $Q = \varphi_{k-1}(\overline{Q})$ is a discrete manifold as well. In brief, with $k > 1$, the only way to have *all* of $T(Q)$ contained in $_kM$ is to have Q discrete.

We pursue the analysis ($k > 1$) directly in the Bowman context. Namely, suppose the regular Lagrangian L is defined as a function from $T(^{k-1}M)$ to $\mathbb{R}$. Let $V = V_L$: $T(^{k-1}M) \to T^2(^{k-1}M)$ be the associated Hamiltonian vector field (defining the equation of motion). Let P be any *subset* in $T(^{k-1}M)$. To say that V is k-suitable over P is to say that, for any differentiable K-curve g in $T(^{k-1}M)$ lying entirely in P with $g' = V \circ g$, g has the form $g = {}_{k-1}I \circ f^{(k)}$, where f lies in M. We say that V is *completely integrable in P* if, for each $s \in K$ and each $p \in P$, there is a differentiable K-curve g in $T(^{k-1}M)$ lying entirely in P, with $g(s) = p$ and with $g' = V \circ g$.

Theorem 3. (1) If $V(P) \subseteq T({}_{k-1}I)(T(^kM))$, then V is k-suitable relative to P and $P \subseteq {}_{k-1}I(^kM)$.

(2) If V is k-suitable and completely integrable relative to P, then $V(P) \subseteq T({}_{k-1}I)(T(^kM))$.

Proof. (1) Let τ: $T^2(^{k-1}M) = T(T(^{k-1}M)) \to T(^{k-1}M)$ denote the tangent bundle projection. We show first that $P \subseteq {}_{k-1}I(^kM)$. Let $p \in P$, and let $x \in T(^{k-1}M)$ with $V(p) = T({}_{k-1}I)(x)$. Then we have

$$p = (\tau \circ V)(p) = \tau(V(p)) = \tau(T({}_{k-1}I)(x)) = {}_{k-1}I(\pi^{k-1}(x))$$

whence $p \in {}_{k-1}I(^kM)$.

Next, let g: $U \to T(^{k-1}M)$ be a differentiable curve lying entirely in P with $g' = V \circ g$. Since V is known to be 1-suitable, $g = (\pi^{k-1} \circ g)'$. Now $\pi^{k-1} \circ g$ is a differentiable curve in ^{k-1}M with $(\pi^{k-1} \circ g)' = g$ entirely in ${}_{k-1}I(^kM)$, since $P \subseteq {}_{k-1}I(^kM)$. Thus, by our $k - 1$ suitability criterion in the Bowman context, $(\pi^{k-1} \circ g) = f^{(k-1)}$, where f lies in M. But then, by definition of $f^{(k)}$, we have ${}_{k-1}I \circ f^{(k)} = (f^{(k-1)})' = (\pi^{k-1} \circ g)' = g$, whence g is a k-suitable curve. Thus, V itself is k-suitable relative to P.

(2) With $p \in P$ and with $s \in K$, consider the initial value problem (V, s, p). By complete integrability, there is a differentiable solution curve g in $T(^{k-1}M)$ lying entirely in P. By k-suitability this curve must have the form $g = {}_{k-1}I \circ f^{(k)}$, where f lies in M. Then we have

$$V(p) = V(g(s)) = g'(s) = ({}_{k-1}I \circ f^{(k)})'(s) = T({}_{k-1}I)((f^{(k)})'(s))$$

whence $V(p) \subseteq T({}_{k-1}I)(T(^kM))$.

Corollary. Let P be a strongly embedded, V-invariant subset/submanifold in $T({}^{k-1}M)$. Then V is completely k-integrable relative to P if and only if $V(P) \subseteq T({}_{k-1}I)(T({}^{k}M))$.

Proof. To say that P is V-invariant is to say that $V(P) \subseteq T(\mathrm{i})(T(P))$, where i: $P \to T({}^{k-1}M)$ is the inclusion map. Since i is a strong embedding, so is $T(\mathrm{i})$, whence V pulls back to a differentiable vector field V_P: $P \to T(P)$. By smoothness alone (the fundamental existence/uniqueness theorem) V_P is completely integrable, whence V is completely integrable relative to P. Then the assertion of the corollary follows immediately from Theorem 3.

Note. Theorem 3 and its corollary apply to *any* 1-suitable V defined over $T({}^{k-1}M)$, whether or not V is derived from a regular Lagrangian function.

Again with $V = V_L$, the Corollary *does not* reveal that the matter of finding a submanifold P as indicated can be formidable. To see the nature of the difficulty in its *general* form, one must work through the local (finite-dimensional) derivation of V_L. Without going through this, however, it is relatively easy to see, for instance, why ($k > 1$) V_L is not necessarily k-suitable relative to $P = {}_{k-1}I({}^{k}M)$ in $T({}^{k-1}M)$.

Namely, let $M = \mathbb{R}$, whence ${}^{k-1}M = \mathbb{R}^k$, $T({}^{k-1}M) = \mathbb{R}^k \times \mathbb{R}^k$, and ${}^{k}M = \mathbb{R}^{k+1}$. Let L: $\mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ be the (kinetic energy) Lagrangian $L(u_0, \ldots, u_{k-1}; v_0, \ldots, v_{k-1}) = \frac{1}{2} \Sigma\, v_j^2$ associated with the standard Riemannian metric over $\mathbb{R}^k$. One can develop V_L through the Legendre transformation (essentially an identity map), the fundamental 1-form on $T^*(\mathbb{R}^k)$, etc., the result of which is that V_L is given by

$$\begin{aligned} &V_L(u_0, \ldots, u_{k-1}, v_0, \ldots, v_{k-1}) \\ &\quad = (u_0, \ldots, u_{k-1}, v_0, \ldots, v_{k-1}; v_0, \ldots, v_{k-1}, 0, \ldots, 0) \end{aligned}$$

It is easy to see that the solutions g to $y' = V_L(y)$ are all functions of the form

$$g(t) = (g_0(t), \ldots, g_{k-1}(t), h_0(t), \ldots, h_{k-1}(t))$$

where $h_i(t) \equiv B_i$ and $g_i(t) = B_i t + A_i$ for $i = 0, \ldots, k-1$. Obviously, $g(t) = f'(t) = (f(t); d/dt[f(t)])$, where $f(t) = (f_0(t), \ldots, f_{k-1}(t))$ with $f_i(t) = B_i t + A_i$. But it is equally obvious, since $k > 1$, that f itself is not generally of the form

$$(e(t), d/dt[e(t)], \ldots, d^{k-1}/dt^{k-1}[e(t)]) = e^{(k-1)}(t)$$

Moreover, the k-suitability criterion established by Theorem 3 (and its corollary) fails dramatically for $P = {}_{k-1}I({}^{k}M)$.

For, by linearity of ${}_{k-1}I$, one easily calculates that

$$\begin{aligned} &T({}_{k-1}I)(T(^kM)) \\ &\quad = \{(r_0, \ldots, r_{k-1}, r_1, \ldots, r_k; s_0, \ldots, s_{k-1}, s_1, \ldots, s_k): \\ &\qquad \text{each } r_i, s_i \in \mathbb{R}\} \end{aligned}$$

It follows that $V_L(u_0, \ldots, u_{k-1}, u_1, \ldots, u_{k-1}, v_{k-1})$ is in $T({}_{k-1}I)(T(^kM))$ if and only if $0 = u_2 = \cdots = u_{k-1} = v_{k-1}$.

On the other hand, suppose we let

$$P = \{(u_0, u_1, 0, \ldots, 0; u_1, 0, \ldots, 0) \in T(^{k-1}M): u_0, u_1 \in \mathbb{R}\}$$

Clearly P is a strongly embedded subset/submanifold in $T(^{k-1}M)$. It is easy to see that $T(P)$, viewed as a subset/submanifold in $T^2(^{k-1}M)$, consists of all

$$(u_0, u_1, 0, \ldots, 0; u_1, 0, \ldots, 0; w_0, w_1, 0, \ldots, 0; w_1, 0, \ldots, 0)$$

where u_0, u_1, w_0, w_1 are in $\mathbb{R}$. Then P is V_L-invariant $[V_L(P) \subseteq T(P)]$ and $V_L(P) \subseteq T({}_{k-1}I)(T(^kM))$, since

$$\begin{aligned} &V_L(u_0, u_1, 0, \ldots, 0; u_1, 0, \ldots, 0) \\ &\quad = (u_0, u_1, 0, \ldots, 0; u_1, 0, \ldots, 0; u_1, 0, \ldots, 0; 0, \ldots, 0) \end{aligned}$$

Thus, by the Corollary, V_L is both completely integrable and k-suitable over P.

We can, of course, verify the foregoing assertion directly. Indeed, consider an initial value problem $(V_L, s, (u_0, u_1, 0, \ldots, 0; u_1, 0, \ldots, 0)$ over P. The solution curve g is given by

$$g(t) = (u_1t + (u_0 - u_1s), u_1, 0, \ldots, 0; u_1, 0, \ldots, 0)$$

Then $g(t) = f'(t) = (f(t); d/dt[f(t)])$, where $f(t) = (u_1t + (u_0 - u_1s), u_1, 0, \ldots, 0)$. Finally, $f(t) = g_0^{(k-1)}(t)$, where $g_0(t) = u_1t + (u_0 - u_1s)$.

Note. In general, P satisfying the terms of the Corollary, when it can be found at all, will not be unique. Nor, absent deeper restrictions on L and M, can one expect P to be determined in some canonical fashion. Nevertheless, specification of an appropriate P is clearly a *prerequisite* to a serious consideration of higher order Lagrangian/Hamiltonian structure.

CONCLUSIONS

1. The essence of Theorems 1 and 2 (and the associated propositions and corollaries) is that the Bowman (restricted) tangential framework, and *not* the full iterated tangential context, is the "correct" one for treatment of higher order differential lift equations. The result follows just from the behavior of successive curve lifts. The fact that the Bowman context captures

every k-suitable higher order equation can be traced directly to the result (Proposition 1) that ${}_kM'$ behaves as though it were the tangent bundle manifold over ${}_kM$.

2. By Example 1, the Bowman context lends itself directly to an interpretive scheme: position, position/velocity, position/velocity/acceleration, etc. Thus, motion problems are readily treatable in this context.

3. As Example 2 shows, extending Lagrangian/Hamiltonian structures to higher differential levels (third-order, fourth-order, etc., differential lift equations) is *not* a simple matter if one wishes to retain the property that solution curves must always arise as differential lifts of curves in the underlying base manifold M. The Corollary to Theorem 3 stands as an unavoidable challenge.

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